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1993 J. Phys. A: Math. Gen. 26 L353

(http://iopscience.iop.org/0305-4470/26/6/010)

J. Phys. A: Math. Gen. 26 (1993) L353–L355. Printed in the UK

LETTER TO THE EDITOR

A free energy bound for the Hopfield model

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Received 4 August 1992

Abstract. We give a simple upper and lower bound on the free energy density of the Hopfield model of size N with p stored patterns, in the limit where N and p tend to infinity with $p/N \to \alpha < 1$. The two bounds coincide for $\alpha = 0$.

The Hamiltonian of the Hopfield model with p stored N-bit patterns is a random function on the set $\mathcal{S} = \{+1, -1\}^N$, with values given by the equation

$$H_{p,N,\xi}(S) = -\frac{1}{2N} \sum_{i,j=1}^{N} \sum_{\mu=1}^{p} \xi_i^\mu \xi_j^\mu S_i S_j \qquad S \in \mathcal{S} \tag{1}$$

where ξ is a $p \times N$ matrix whose elements ξ_i^μ are independent random variables with values ± 1 and mean zero. The quantity considered here is the expected value of the free energy density

$$f_{p,N,\beta}(\xi) = -\frac{1}{\beta N} \ln \left[\sum_{S \in \mathcal{S}} e^{-\beta H_{p,N,\xi}(S)} \right] \tag{2}$$

at positive inverse temperatures β. In order to state our result, we define

$$\phi_\delta(\beta, h) = -\frac{1}{\beta} \ln [2 \cosh(\beta h)] + \frac{1-\delta}{2} h^2. \tag{3}$$

Theorem. Let $0 \leqslant \alpha < 1$ and $\delta < 1$ such that $\delta - 4\sqrt{\alpha}(1-\delta)$ is positive. Then for any $\beta > 0$

$$\min_{h \in \mathbb{R}} \phi_\delta(\beta, h) + \frac{\alpha}{2\beta} \ln[\delta - 4\sqrt{\alpha}(1-\delta)] \leqslant \lim_{\substack{N,p \to \infty \\ p/N \to \alpha}} \mathbb{E} f_{p,N,\beta}(\xi) \leqslant \min_{h \in \mathbb{R}} \phi_0(\beta, h) \tag{4}$$

where 'lim' stands for either 'lim sup' or 'lim inf'.

Remarks. The upper bound in (4) is well known: it is the free energy density of the Curie–Weiss model, which is obtained by dropping all terms with $\mu > 1$ from the Hamiltonian (1). For $\alpha = 0$ the lower bound in (4) coincides with the upper bound as $\delta \downarrow 0$. In this case we recover a recent result of Shcherbina and Tirozzi [2]. For $\beta < 1$, the lower bound with $\delta = 1 - \beta$ agrees with the correct free energy density up to $\mathcal{O}(\alpha^{3/2})$; see e.g. the references [1] and [3].

† Supported in part by the National Science Foundation under Grant No. DMS-9103590.

Proof. Consider the overlap matrix A,

$$A_{\mu\nu}=\frac{1}{N}\sum_{i=1}^{N}\xi_i^{\mu}\xi_i^{\nu}-\delta_{\mu\nu}\qquad \mu,\nu=1,\ldots,p. \tag{5}$$

An estimate in [2] implies that there exists a real number λ and an even positive integer n, both depending on N, such that

$$\lambda^{-n}\mathbb{E}\operatorname{tr}A^n\leqslant o(N^{-1})\qquad \lambda=4\sqrt{\alpha}+o(1) \tag{6}$$

as $N\to\infty$. A stronger version of this estimate (inequality (11)), which could be of independent interest, will be proved below.

Assume now that α and δ satisfy the hypothesis of the theorem, and that N is sufficiently large such that $\delta-\lambda(1-\delta)>0$. Denote by $\langle\cdot,\cdot\rangle$ the standard inner product on $\mathbb{R}^p$, and denote by ξ_i the ith column of the matrix ξ. If ξ is such that the operator norm of A is less than or equal to λ, we get the following upper bound on the partition function:

$$\begin{aligned}
&\sum_{S\in\mathcal{S}}\exp[-\beta H_{p,N,\xi}(S)]\\
&\quad=\left(\frac{N\beta}{2\pi}\right)^{p/2}\int \mathrm{d}^p m\,\exp[-\tfrac{1}{2}N\beta\langle m,m\rangle]\sum_{S\in\mathcal{S}}\exp\left[\beta\sum_{i=1}^{N}\langle m,\xi_i\rangle S_i\right]\\
&\quad=\left(\frac{N\beta}{2\pi}\right)^{p/2}\int \mathrm{d}^p m\,\exp[-\tfrac{1}{2}N\beta\langle m,m\rangle]\prod_{i=1}^{N}2\cosh(\beta\langle m,\xi_i\rangle)\\
&\quad=\left(\frac{N\beta}{2\pi}\right)^{p/2}\int \mathrm{d}^p m\,\exp[-\tfrac{1}{2}N\beta\langle m,[\delta+(1-\delta)A]m\rangle-\beta]\sum_{i=1}^{N}\phi_\delta(\beta,\langle m,\xi_i\rangle)\\
&\quad\leqslant\left(\frac{N\beta}{2\pi}\right)^{p/2}\int \mathrm{d}^p m\,\exp[-\tfrac{1}{2}N\beta\langle m,[\delta-\lambda(1-\delta)]m\rangle]\max_{h\in\mathbb{R}}\exp[-\beta N\phi_\delta(\beta,h)]\\
&\quad=[\delta-\lambda(1-\delta)]^{-p/2}\exp\left[-\beta N\min_{h\in\mathbb{R}}\phi_\delta(\beta,h)\right].
\end{aligned}\tag{7}$$

Now we take the logarithm and divide by $-\beta N$. The resulting lower bound on the free energy density (2) can be extended to all patterns ξ, by adding to it the trivial bound $-[p/2+\text{const}]$, multiplied by the positive factor $\lambda^{-n}\operatorname{tr}A^n$ which is larger than 1 whenever $\|A\|>\lambda$:

$$f_{p,N}(\beta,\xi)\geqslant\min_{h\in\mathbb{R}}\phi_\delta(\beta,h)+\frac{p}{2\beta N}\ln[\delta-\lambda(1-\delta)]-[p/2+\text{const}]\lambda^{-n}\operatorname{tr}A^n. \tag{8}$$

The assertion now follows from (8) and (6).

To complete the proof of our theorem we will now derive a bound that implies (6). Let $n\geqslant 4$. Then $N^n\mathbb{E}\operatorname{tr}A^n$ is equal to the number of ordered products

$$\Pi_0\equiv\xi_{i_1}^{\mu_1}\xi_{i_1}^{\mu_2}\xi_{i_2}^{\mu_2}\xi_{i_2}^{\mu_3}\ldots\xi_{i_{k-1}}^{\mu_k}\{\xi_{i_k}^{\mu_k}\ldots\xi_{i_l}^{\mu_l}\}\xi_{i_l}^{\mu_{l+1}}\ldots\xi_{i_n}^{\mu_n}\xi_{i_n}^{\mu_1} \tag{9}$$

(ignoring the braces) with $1\leqslant\mu_k\leqslant p$ and $1\leqslant i_k\leqslant N$ for all k, subject to the constraint that $\mu_1\neq\mu_2\neq\ldots\neq\mu_n\neq\mu_1$, and that each random variable ξ_i^μ appears an even number of times. Such products will be referred to as *contributing* products. Note that the constraint implies that the cardinalities $u=|\{\mu_1,\ldots,\mu_n\}|$ and $v=|\{i_1,\ldots,i_n\}|$ satisfy $u+v\leqslant n+1$ and $2v\leqslant n$.

A contributing product will be called *simple* if its $2n$ factors can be grouped into n pairs of identical random variables in such a way that if each pair is marked by braces as shown in equation (9) for the pair $(\xi_{i_k}^{\mu_k}, \xi_{i_l}^{\mu_l})$, then the 'sets' corresponding to different pairs are either nested or disjoint. Note that such pairings can be specified by giving only the n left braces. Thus, since each pairing determines whether or not any two indices can have different values, the number of simple products can be bounded by

$$c_n(p) \equiv \binom{2n}{n} \max_{\substack{u+v \leqslant n+1 \\ v \leqslant n/2}} p^u N^v. \tag{10}$$

Assume now that Π_0 is a non-simple contributing product. Below we will give a procedure for cutting Π_0 between some top-linked pairs (adjacent factors that have a common upper index, or the first and last factor) and regrouping the pieces into a simple product Π_m that has the following property P: the first factor of Π_m is the same as that of Π_0; and if ω is any closed walk on the factors of Π_m such that every odd-numbered (even-numbered) step is between two factors that are top-linked in Π_m (Π_0), then, with one possible exception, all pairs connected by an odd-numbered step of ω are pairs of identical factors. In particular, if we only consider walks whose first step goes to the right, and for which all but the last (or all) odd-numbered steps are between identical pairs, then we can find a set W of walks of this type, such that every factor of Π_m is visited by exactly one walk in W, and only once. Thus, it is possible to encode the original product Π_0 in a modified version Π_m' of Π_m, where, along each walk $(\omega_0, \omega_1, \ldots, \omega_l = \omega_0) \in W$ of length $l \geqslant 4$, the upper index in the identical factors ω_{2k-2} and ω_{2k-1} has been replaced by a pointer to the factor ω_{2k}, for $1 \leqslant k < l/2$. Since the upper indices of Π_m' can take at most $p+2n$ different 'values', it follows that the number of contributing products is bounded by $c_n(p+2n)$, and hence

$$\mathbb{E}\,\mathrm{tr}\, A^n \leqslant N^{-n} c_n(p+2n) \leqslant 4^n (p+2n) \left(\frac{p+2n}{N}\right)^{n/2} \qquad p+2n \leqslant N. \tag{11}$$

We get (6) by taking, e.g. $\lambda = 4(1+n^{-1/2})((p+2n/N)^{1/2}$ with $n =$ twice the integer part of $\sqrt{N}$.

Finally, to construct Π_m from Π_0 we iterate the following step. Assume that Π_{m-1} is non-simple. Then, in this product, it is possible to find an odd number $\geqslant 3$ of consecutive factors $\xi_{i'}^{\mu}\xi_i^{\mu} \ldots \xi_{j'}^{\mu}$ such that $\xi_{j'}^{\mu}$ is top-linked to a factor ξ_j^{μ} that is identical to $\xi_{i'}^{\mu}$, with $i \neq i'$ and $j \neq j'$. After choosing such a sub-product, we now reverse the order of the factors $\xi_i^{\mu} \ldots \xi_{j'}^{\mu}$ in Π_{m-1} and define Π_m to be the result of this operation. Note that this creates a new top-linked pair $(\xi_{i'}^{\mu}, \xi_{j'}^{\mu})$ of identical factors. Thus, the iteration ends (with a simple product) after less than n steps. It is easy to check that each step preserves the abovementioned property P, and that contributing products are mapped to contributing products.

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